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Abatement, consumption, capital, and pollution accumulation in an optimal programme

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The authors develop an optimal control model in which abatement and consumption are chosen to maximize the present discounted value of utility subject to the dynamical system governing the accumulation of the stocks of capital and pollution. The steady-state comparative statics and welfare effects of technology and preference shocks are derived and discussed. The strength of the model lies in its separation of the consumption and abatement choices, which best reflects the trade-offs faced by policy-makers, and alters or expands some of the conclusions reached by previous researchers who have studied problems of less generality.

Introduction

Some empirical evidence on the relationship between economic growth and the environment is starting to mount. One of the more in-depth studies in this area is the recent paper by Grossman, Krueger (1995). In order to study the relationship between pollution and growth, they estimated reduced-form equations that relate per capita income to various environmental indicators. They concluded 'contrary to the alarmist cries of some environmental groups, we find no evidence that economic growth does unavoidable harm to the natural habitat. Instead, we find that while increases in gross domestic product (GDP) may be associated with worsening environmental conditions in very poor countries, air and water quality appear to benefit from economic growth, once some critical level of income has been reached'. They also pointed to the consistency of their empirical findings with those of others. What is most relevant for our purposes is that, in instances where economic growth has been associated with environmental improvement, they believe there is little evidence to suggest that it has been an automatic one. Rather, such countries that have grown and reduced pollution have engaged in an *active process of abatement*. It is exactly this empirical feature that our optimal growth model is designed to capture: the ability of a country to both create and abate pollution in an optimal plan.

In order to propose a solution to environmental problems one must first understand the fundamental economic forces and trade-offs that shape a country's choices of abatement, consumption, and capital and pollution accumulation. One of the first theoretical papers in economics that investigated such a consumption–pollution trade-off is by Forster (1973a). He ignored growth considerations, and thus assumed that output in each period was fixed and constant over time. In this manner, Forster (1973a) simplified the optimal control problem, so that it was only necessary to choose the time-path of consumption, for, once it was determined, abatement was computed residually from the exogeneity of output. He, therefore, did not allow society to choose its consumption and abatement rates of the pollution stock independently, something that we allow for in our model.

Closer in spirit to our research is another paper by Forster (1973b). His main concern was investigating the effect of adding a flow pollutant into the neo-classical optimal growth model. By treating pollution as a flow which enters the instantaneous utility function, he avoided introducing a second-state variable, thereby permitting direct comparison with the archetype neo-classical optimal growth model. Since output is not fixed in the neo-classical optimal growth model, abatement and consumption are chosen as independent control variables analogous to our model. His main conclusion was that, in the presence of a flow pollutant, the optimal steady-state levels of capital and consumption are lower than in the neo-classical prototype. Unfortunately, Forster (1973b) did not investigate the comparative statics properties of the steady state of his model, nor did he consider stock pollutants, both of which are considered in this paper.

In a more recent and comprehensive paper, van der Ploeg, Withagen (1991) presented a smorgasbord of optimal growth and pollution control models and discussed the similarities and difference of their solutions. Most relevant to our works is their pollution control and optimal growth model of Section 4 of their paper, which they discuss later in Section 7 as well. This model of theirs differs from ours in three important ways: (i) they assumed that the flow of pollution is a function of output and thus the capital stock, while we assume it to be a function of consumption; (ii) they assumed that the decay rate of the pollution stock is a function of the abatement rate, while we assume it to be a constant; and (iii) they assumed that the abatement rate affects the rate of change of pollution stock only through the decay rate, while we assume that it affects the rate of change of the pollution stock independently of nature's own cleansing ability. They showed that if the discount rate is small enough, then the steady state is a local saddlepoint, but concluded that the steady-state 'comparative statics calculations are rather tedious and do not lead to unambiguous outcomes.' This quote is in sharp contrast to the results obtained here, where over one-half of the steady-state comparative statics are unambiguous. Moreover, we consider a more comprehensive set of parameter shocks to the steady state and include a discussion of their effects on society's welfare.

Our contribution to the literature is thus four-fold: (i) we explicitly allow for consumption and abatement to be independent decision variables, and thus model the more realistic capital accumulation and pollution accumulation trade-off; (ii) we investigate the local stability properties of the steady state in a more general model; (iii) we conduct a comparative statics investigation of the steady state, considering perturbations in preference and technology parameters and the social rate of discount; and (iv) we investigate the welfare effects of perturbations in preference and technology parameters. By allowing for part of the aggregate output of the economy to be used for the pollution abatement, we capture the essential trade-off between capital investment and abatement. All else the same, capital investment leads to higher future output and consumption, but also to a higher stock of pollution. In contrast, abatement expenditures lead to a lower stock of pollution, but at the cost of lower future output and consumption, *ceterus paribus*. Moreover, by modelling such a trade-off, we account for the types of choices encountered by decision makers that earlier research has ignored.

Economic interpretation of the model

Consider an economy in which output Y is a function of the capital stock K and the technological efficiency of production α_1, say $Y = f(K;\alpha_1)$. In each time-period, society allocates its income Y between consumption C, gross investment I, and pollution abatement A; therefore $Y = C + A + I$. Gross investment can be sub-divided into net capital investment $\dot{K}$ and replacement $\delta_1 K$, where $\delta_1 > 0$ is the depreciation rate of capital. The breakdown of income can, therefore, be written as $f(K;\alpha_1) = C + \dot{K} + A + \delta_1 K$. Pollution abatement is defined as income spent to clean up the pollution stock.

The state of the economy is described not only by the stock of capital, but also by the stock of pollution present at time t. Consumption contributes directly to the accumulation of pollution, while abatement and nature's own cleansing ability, $\delta_2 A$, lower the rate at which the pollution stock accumulates, where $\delta_2 > 0$ is the natural decay rate of the pollution stock. Since, in general, both production and consumption are activities that lead to the accumulation of pollution, our model employs the simplifying assumption that consumption is the only polluting factor. While this is less general than allowing the rate of accumulation of the pollution stock to depend on output and consumption, such a simplifying assumption is employed because it sharpens the economic intuition and insight obtained from the model. Moreover, it simplifies the mathematical analysis of the relatively complicated two-state optimal control problem, which is a class of control problems from which it is notoriously difficult to extract qualitative information.

Along an optimal path, society will act to maximize the present discounted value of instantaneous utility given the social discount rate $r > 0$. Instantaneous utility U is affected directly and positively by consumption, but negatively by the stock of pollution present in the environment. Utility is also indirectly affected by society's choices of consumption, abatement, and investment in capital. Consumption lowers utility indirectly through its impact on the accumulation of the pollution stock, and through reduced capital accumulation and, hence, lower future output and consumption. Abatement also has two indirect effects on utility, one positive, through reduced pollution levels, and one negative through reduced capital accumulation, and, therefore, lower future output and consumption. Similarly, capital investment has a positive indirect effect on utility due to its positive effect on future production, and, hence, consumption, but it also negatively affects utility through its indirect effect on the pollution stock through the increase in future output and consumption.

The socially optimal time-paths of abatement and consumption are, therefore, defined to be the solution to the following intertemporal maximization problem:

$$\text{(P)} \qquad V(K_0, P_0, \beta) := \max_{A(t), C(t)} \int_0^{+\infty} U[C(t), P(t); \alpha_2, \alpha_3] e^{-rt} dt$$

$$\text{s.t.} \quad \dot{K}(t) = f[K(t);\alpha_1] - C(t) - A(t) - \delta_1 K(t), \; K(0) = K_0$$

$$\dot{P}(t) = g[C(t);\alpha_4] - h[A(t);\alpha_5] - \delta_2 P(t), \; P(0) = P_0$$

$$A(t) \geq 0, \; C(t) \geq 0, \; K(t) \geq 0, \; P(t) \geq 0$$

where $\beta := (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \delta_1, \delta_2, r)$ denotes a vector of time-independent exogenous parameters of the problem. The parameters (α_2, α_3) are preference shifters, while $(\alpha_1, \alpha_4, \alpha_5)$ are technology shifters. Their economic interpretations will be made more transparent when the assumptions of the model are discussed and the steady-state comparative statics are presented.

The following assumptions are placed on the model:

$U: \Re^2_+ \times \Re^2 \to \Re$, $f: \Re_+ \times \Re \to \Re_+$, $g: \Re_+ \times \Re \to \Re_+$, $h: \Re_+ \times \Re \to \Re_+$, and the functions are of class $C^{(2)}$ on their domains. (A.1)

$$U_c > 0,\ U_{\alpha_2} > 0,\ U_{CC} < 0,\ U_{CP} \equiv 0,\ U_{C\alpha_2} > 0,\ U_{C\alpha_3} \equiv 0,$$

$$U_P < 0,\ U_{\alpha_3} > 0,\ U_{PP} < 0,\ U_{P\alpha_2} \equiv 0,\ U_{P\alpha_3} > 0 \tag{A.2}$$

$$f(0;\alpha_1) = 0,\ f_K > 0,\ f_{KK} < 0,\ f_{\alpha_1} > 0,\ f_{K\alpha_1} > 0 \tag{A.3}$$

$$g(0;\alpha_4) = 0,\ g_C > 0,\ g_{CC} > 0,\ g_{\alpha_4} < 0,\ g_{C\alpha_4} < 0 \tag{A.4}$$

$$h(0;\alpha_5) = 0,\ h_A > 0,\ h_{AA} < 0,\ h_{\alpha_5} > 0,\ h_{A\alpha_5} > 0 \tag{A.5}$$

$\exists$ a bounded interior solution to the necessary conditions of the optimal control problem (P) $\forall\theta \in B(\theta^o; \varepsilon_1)$, where $\theta := (K_0, P_0, \beta)$ and $\theta^0 \in \Re^{10}_{++}$ is a fixed value of θ, denoted by the vector $(A,C,K,P,\lambda_1,\lambda_2) = [A^o(t;\theta), C^o(t;\theta), K^o(t;\theta), P^o(t;\theta), \lambda_1^o(t;\theta), \lambda_2^o(t;\theta)]$, where λ_1 and λ_2 are the current value costate variables corresponding to the state variables K and P, respectively. (A.6)

The bounded interior solution to the necessary conditions of (P) converges to an interior and simple steady-state solution of the necessary conditions as $t \to +\infty$, the latter of which is denoted by the vector $[A^*(\beta), C^*(\beta), K^*(\beta), P^*(\beta), \lambda_1^*(\beta), \lambda_2^*(\beta)]$, which exists $\forall\beta \in B(\beta^o; \varepsilon_2)$, where $\beta^o \in \Re^8_{++}$. (A.7)

Assumption (A.1) defines the functions as elements of the set of twice continuously differentiable functions, a standard assumption when the differential calculus is employed as the tool of analysis and the focus is on the qualitative properties of the model, as it is here. Assumption (A.2) asserts that the instantaneous utility function is increasing in consumption but at a decreasing rate, i.e., there is positive but decreasing marginal utility of consumption. It also asserts that utility decreases at an increasing rate with the pollution stock. That is, each additional unit of pollution present in the environment causes a greater loss in utility than did the unit before it. This is a logical assumption since, as pollution increases, a clean environment (a scarce resource) becomes increasingly valuable. Thus, each additional unit of pollution will cause a greater loss in utility than did the previous one. The marginal utility of consumption is assumed to be independent of the pollution stock. This simplifying assumption is made for the same reasons that were given in the above discussion of the differential equation governing the accumulation of the pollution stock. In addition, an increase in the parameter α_2 is assumed to increase the marginal utility of consumption, while an increase in the parameter α_3 is assumed to lower the marginal disutility of the pollution stock.

Assumption (A.3) asserts that the capital stock is essential to production and exhibits a positive but declining marginal product. Furthermore, as the parameter α_1 increases, so too does the marginal product of capital and output. Assumption (A.4) states that the rate of pollution accumulation is an increasing strongly convex function of society's consumption rate, and the pollution does not accumulate if consumption is zero. In addition, an increase in α_4 decreases the marginal effect that consumption has on pollution accumulation. Assumption (A.5) asserts that the marginal product of abatement is positive but declining (i.e., less pollution is removed at the margin for each additional unit of abatement), and that no pollution is removed when abatement is zero. Moreover, an increase in the parameter α_5 increases the marginal product of abatement. Assumptions (A.6) and (A.7) state that a bounded interior solution to the necessary conditions of the control problem exists which converges to the steady state of the problem. Since consumption, pollution, capital, and abatement are bounded positive policy variables in an aggregate economy, these assumptions align our model with realistic empirical features and eliminate the need for us to dwell on tangential mathematical details which are unimportant, given our qualitative focus in the paper. Finally, note that a simple steady state is, by definition, one in which the Jacobian matrix (11) of its associated dynamical system (10) has a non-zero determinant.

Define the current value Hamiltonian as

$$\begin{aligned} H(A,C,K,P,\lambda_1,\lambda_2;\beta) := {} & U(C,P;\alpha_2,\alpha_3) + \lambda_1[f(K;\alpha_1) - C - A - \delta_1 K] \\ & + \lambda_2[g(C;\alpha_4) - h(A;\alpha_5) - \delta_2 P] \end{aligned} \tag{1}$$

By Theorem 3.12 of Seierstad, Sydsaeter (1987: 234), the necessary conditions include:

$$H_C = U_C(C;\alpha_2) - \lambda_1 + \lambda_2 g_C(C;\alpha_4) = 0 \tag{2a}$$

$$H_A = -\lambda_1 - \lambda_2 h_A(A;\alpha_5) = 0 \tag{2b}$$

$$\dot{\lambda}_1 = r\lambda_1 - H_K = r\lambda_1 - \lambda_1[f_k(K;\alpha_1) - \delta_1] \tag{2c}$$

$$\dot{\lambda}_2 = r\lambda_2 - H_P = (r + \delta_2)\lambda_2 - U_P(P;\alpha_3) \tag{2d}$$

$$\dot{K} = H_{\lambda_1} = f(K;\alpha_1) - C - A - \delta_1 K \tag{2e}$$

$$\dot{P} = H_{\lambda_2} = g(C;\alpha_4) - h(A;\alpha_5) - \delta_2 P. \tag{2f}$$

Before discussing the implications of the necessary conditions, we will show that the bounded interior solution to the necessary conditions that converges to the steady-state solution of the necessary conditions is the unique solution to the control problem under a mild additional assumption. First, recall that (i) U is a strongly concave function of (C,P) by (A.2); (ii) $\lambda_1 f$ is a strongly concave function of K by (A.3) and the fact that $\lambda_1 > 0$ along an optimal path by (A.2), (A.5), (5), and (6); (iii) $\lambda_2 g$ is a strongly concave function of C by (A.4) and the fact that $\lambda_2 < 0$ along an optimal path by (A.2) and (5). Thus, the Hamiltonian is a sum of additively separable strongly concave and linear functions of the state and control variables. It, therefore, follows by Theorem 1.E.11(ii) of Takayama (1985) and a straightforward calculation that the Hessian matrix of the Hamiltonian with respect to the state and control variables is negative definite. Hence, by Theorem 1.E.12(ii) of Takayama (1985), the Hamiltonian is a strictly concave function of the state and control variables. Moreover, assuming that the admissible values of the state variables possess a limit as $t \to +\infty$, the conditions of Theorem 3.13 of Seierstad, Sydsaeter (1987: 234) are satisfied. Hence, we can conclude that the bounded interior solution to the necessary conditions that converges to the steady-state solution of the necessary conditions is the unique solution to the optimal control problem (P).

We commence the analytical discussion of the necessary and sufficient conditions with Equation (2d), a first-order differential equation in λ_2 with integrating factor $\mu(t) := e^{-(r+\delta_2)t}$ Multiplying both sides of (2d) by $\mu(t)$ yields

$$\frac{d}{dt}\left[e^{-(r+\delta_2)t}\lambda_2\right] = -e^{-(r+\delta_2)t}U_P[P(t);\alpha_3]. \tag{3}$$

Integrating both sides of (3) with respect to t and dividing both sides by $e^{-(r+\delta_2)t}$ gives

$$\lambda_2(t) = \int_t^{+\infty} e^{-(r+\delta_2)(s-t)}U_P(P(s);\alpha_3)ds + ae^{(r+\delta_2)t} \tag{4}$$

as the general solution to (2d), where s is the dummy variable of integration and a is a constant of integration. Since $\lambda_2(t)$ is the current value shadow cost of pollution at time t, we are concerned with evaluating the integral from the present time t to infinity.

By (A.6), (A.7), and (4),

$$\lim_{t\to+\infty} \lambda_2(t) = \lim_{t\to+\infty} \left\{ \int_t^{+\infty} e^{-(r+\delta_2)(s-t)} U_P[P(s);\alpha_3]ds + ae^{(r+\delta_2)t} \right\} \text{exists.}$$

As U_P is bounded along the optimal path by (A.1) and (A.6), and since $(r + \delta_2)(s - t)$ is non-negative by the definition of the integral and the assumptions that the social rate of discount and the natural decay rate of the pollution stock are positive, the limit of the integral exists. But since $e^{(r+\delta_2)t} \to +\infty$ as $t \to +\infty$, must be equal to zero in order for $\lim_{t\to+\infty} \lambda_2(t)$ to exist. Hence the general solution (4) reduces to the following specific solution for the current value shadow cost of pollution:

$$\lambda_2(t) = \int_t^{+\infty} e^{-(r+\delta_2)(s-t)} U_P[P(s);\alpha_3]ds. \tag{5}$$

This equation states that under optimal conditions the current value shadow cost of pollution is equal to the current value of all utility lost due to the pollution in the environment from the present time onward, discounted at the social discount adjusted for the natural decay rate of the pollution stock. Since $U_P < 0$ by (A.2), $\lambda_2(t) < 0 \; \forall t \in (0, +\infty)$ in an optimal plan, exactly as one would expect since the stock of pollution is bad.

Equation (2b) can be rewritten as

$$-\lambda_2 h_A(A;\alpha_5) = \lambda_1. \tag{6}$$

As $\lambda_2 < 0$ as noted above and $h_A > 0$ by (A.5), (6) implies that the current value shadow price of the capital stock is positive in an optimal plan, i.e., $\lambda_1(t) > 0 \; \forall t \in (0, +\infty)$. Equation (6) asserts that in an optimal programme, society will choose to abate pollution until the marginal value of abatement (in terms of pollution reduction) is equal to the current value shadow price of capital. At this point, there is no incentive for society to switch resources from abatement to capital investment, or vice versa. This equation has no counterpart in the prototype optimal growth and pollution control models.

From (2a) we see that

$$U_C(C;\alpha_2) = \lambda_1 - \lambda_2 g_C(C;\alpha_4). \tag{7}$$

Equation (7) asserts that along the optimal path society will choose to consume at the point where the marginal utility of consumption is equal to the true marginal cost of consumption, the latter of which is comprised of the foregone utility due to consuming the marginal unit of output rather than investing it in the capital stock (λ_1), and the utility decrease due to the pollution caused by consuming the marginal unit of output. Again, we see that when society is operating along with optimal path, there is no incentive to reallocate resources between consumption and investment in capital. More importantly, Equation (7) shows that, *ceteris paribus*, the optimal consumption rate in a model that accounts for capital, and pollution accumulation is smaller than if only one of these stocks is accounted for.

Equation (2e) is a first-order differential equation describing the change in the current value shadow price of capital over time. Rearranging the terms yields

$$\dot{\lambda}_1 = (r + \delta_1)\lambda_1 - \lambda_1 f_K(K;\alpha_1). \tag{8}$$

The first grouping in Equation (8) is the current value shadow rental cost of capital at time t. Since $f_K(K;\alpha_1)$ is the marginal product of capital, $\lambda_1 f_K(K;\alpha_1)$ is the marginal benefit from investing in a unit of capital. Thus, Equation (8) represents the difference between the shadow rental cost and the marginal benefit of capital at any point in time. Therefore, at any t, there exists three possible cases for the comparative values of the shadow rental cost and marginal benefit of capital: the shadow rental cost of capital is greater than, less than, or equal to the marginal benefit of capital.

If $(r + \delta_1) > f_K(K;\alpha_1)$, then the current value shadow price of capital is increasing over time. Since the marginal product of capital is decreasing by (A.3), this implies that in order to reach the steady-state stock of capital, society must disinvest in the capital stock. If $(r + \delta_1) = f_K(K;\alpha_1)$, then the current value shadow price of capital is constant over time and the capital stock is in a steady state, therefore, society will choose to maintain the current amount of the capital stock. Finally, if $(r + \delta_1) < f_K(K;\alpha_1)$, then the current value shadow price of capital is decreasing over time. Analogous to the story above, it is now optimal for society to invest in the capital stock in order for it to reach its steady-state level.

Stability of the steady-state equilibrium

As our optimal control model includes two state variables, it, therefore, requires the use of a four-dimensional phase diagram to graphically depict the dynamics. As a result, we cannot use a phase portrait to provide a qualitative description of the optimal paths and their approach to the steady state. We, therefore, follow the discussion of local stability from Dockner (1985) in order to gain insight into the qualitative characteristics of the optimal solution. To that end, first observe that the necessary and sufficient condition (2a) is independent of A and (2b) is independent of C. Thus, since $H_{CC} = U_{CC} + \lambda_2 g_{CC} < 0$ by (A.2), (A.4), and $\lambda_2 < 0$, the implicit function theorem and (A.1) imply that (2a) defines C as a locally $C^{(1)}$ function of $(\lambda_1, \lambda_2; \alpha_2, \alpha_4)$. Similarly, because $H_{AA} = -\lambda_2 h_{AA} < 0$ by (A.5) and $\lambda_2 < 0$, the implicit function theorem and (A.1) imply that (2b) defines A as a locally $C^{(1)}$ function of $(\lambda_1, \lambda_2; \alpha_5)$. These short-run optimizing functions are denoted by

$$C = \hat{C}(\lambda_1, \lambda_2; \alpha_2, \alpha_4) \tag{9a}$$

$$A = \hat{A}(\lambda_1, \lambda_2; \alpha_5) \tag{9b}$$

The comparative statics of these functions are given in Appendix I and are important inputs to the stability analysis and the steady-state comparative statics. Substituting Equation (9) into the state and canonical differential equations (2c)–(2f) yields the modified Hamiltonian Dynamical System:

$$\dot{K} = f(K;\alpha_1) - \hat{C}(\lambda_1, \lambda_2; \alpha_2, \alpha_4) - \hat{A}(\lambda_1, \lambda_2; \alpha_5) - \delta_1 K \tag{10a}$$

$$\dot{P} = g[\hat{C}(\lambda_1, \lambda_2; \alpha_2, \alpha_4); \alpha_4] - h[\hat{A}(\lambda_1, \lambda_2; \alpha_5);\alpha_5] - \delta_2 P \tag{10b}$$

$$\dot{\lambda}_1 = \lambda_1[r + \delta_1 - f_K(K;\alpha_1)] \tag{10c}$$

$$\dot{\lambda}_2 = (r + \delta_2)\lambda_2 - U_P(P;\alpha_3). \tag{10d}$$

The steady state of Equation (10) is found by setting $\dot{K} = \dot{P} = \dot{\lambda}_1 = \dot{\lambda}_2 = 0$ and solving for the state and costate variables in terms of the parameters β. The implicit function theorem guarantees that this can be done locally and will yield steady-state solutions that are $C^{(1)}$ functions of the parameters as long as the Jacobian matrix of (10), defined by

$$J := \begin{bmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial P} & \frac{\partial \dot{K}}{\partial \lambda_1} & \frac{\partial \dot{K}}{\partial \lambda_2} \\ \frac{\partial \dot{P}}{\partial K} & \frac{\partial \dot{P}}{\partial P} & \frac{\partial \dot{P}}{\partial \lambda_1} & \frac{\partial \dot{P}}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_1}{\partial K} & \frac{\partial \dot{\lambda}_1}{\partial P} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_2}{\partial K} & \frac{\partial \dot{\lambda}_2}{\partial P} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} \end{bmatrix}, \qquad (11)$$

has a non-zero determinant when evaluated at the steady state. Because the steady state is simple by (A.7), it follows from the definition of simplicity that there are no zero eigenvalues in the spectrum of J, hence $|J| \neq 0$. The specific formulae for the elements of J are given in Appendix II. Since $|J| \neq 0$, the well-defined steady-state solution for the state and costate variables is

$$K = K^*(\beta) \qquad (12a)$$

$$P = P^*(\beta) \qquad (12b)$$

$$\lambda_1 = \lambda_1^*(\beta) \qquad (12c)$$

$$\lambda_2 = \lambda_2^*(\beta). \qquad (12d)$$

The steady-state values of the control variables are then found by substituting the steady-state values of the state and costate variables given by Equation (12) into Equation (9) to yield

$$C = C^*(\beta) := \hat{C}[\lambda_1^*(\beta), \lambda_2^*(\beta); \alpha_2, \alpha_4] \qquad (13a)$$

$$A = A^*(\beta) := \hat{A}[\lambda_1^*(\beta), \lambda_2^*(\beta); \alpha_5]. \qquad (13b)$$

The determinant of J reduces to the following expression using the results of Appendix II:

$$|J| = \frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial P}\left[\frac{\partial \dot{K}}{\partial \lambda_1}\frac{\partial \dot{P}}{\partial \lambda_2} - \frac{\partial \dot{K}}{\partial \lambda_2}\frac{\partial \dot{P}}{\partial \lambda_1}\right] - \frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_1}\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}. \qquad (14)$$

In general, given assumptions (A.1)–(A.7) and the assertion of dynamic optimization, we are unable to unambiguously sign $|J|$. Therefore, by Theorems 3 and 4 of Dockner (1985: 96, 97), two conditions may hold for the local stability of the steady-state solution to our optimal control problem: either we have conditional stability with a local saddlepoint property or conditional stability with a local one-dimensional stable manifold. It is important to note that for the economy to reach the steady state from an initial state that does not coincide with the steady state, as the optimal solution of (P) does, the steady state must lie on a stable manifold. In other words, the steady state must at least be conditionally stable for problem (P) to be consistent with the assumptions imposed on it.

From Lemma 1 of Dockner (1985: 93), we define

$$\Omega := \begin{vmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial \lambda_1} \\ \frac{\partial \dot{\lambda}_1}{\partial K} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{P}}{\partial P} & \frac{\partial \dot{P}}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_2}{\partial P} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} \end{vmatrix} + 2\begin{vmatrix} \frac{\partial \dot{K}}{\partial P} & \frac{\partial \dot{K}}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_1}{\partial P} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} \end{vmatrix} = \left[-\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_1}\right] + \left[\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} - \frac{\partial \dot{P}}{\partial \lambda_2}\frac{\partial \dot{\lambda}_2}{\partial P}\right] < 0, \qquad (15)$$

the sign of which follows from the results of Appendix II. If $|J| > 0$, then by the Theorem in Tahvonen (1991; Appendix), the sufficient conditions for the steady state to be locally saddlepoint stable are satisfied. If, however, $|J| < 0$, then by Theorem 4 of Dockner (1985: 97), the necessary and sufficient condition for local one-dimensional stability of the steady state is satisfied. By defining $\Psi := (\partial \dot{K}/\partial \lambda_1)(\partial \dot{P}/\partial \lambda_2) - (\partial \dot{K}/\partial \lambda_2)(\partial \dot{P}/\partial \lambda_1)$, it follows from Equation (14) and Appendix II that $\Psi \geq 0$ is a sufficient condition for the steady state to exhibit local saddlepoint stability. We will, therefore, assume that $\Psi \geq 0$ in the evaluation of the steady-state comparative statics of the ensuing section.

The local saddlepoint stability of the steady-state solution can be given a geometric interpretation, even though, as noted above, it is not possible to draw the corresponding four-dimensional phase portrait. In the four-dimensional phase-space defined by the modified Hamiltonian dynamical system (10), the steady-state solution lies at the intersection of the four corresponding isoclines, $\dot{K} = 0$, $\dot{P} = 0$, $\dot{\lambda}_1 = 0$, and $\dot{\lambda}_2 = 0$. Under the sufficient condition $\Psi \geq 0$, the steady state is a saddlepoint, so that two eigenvalues of J have negative real parts and two eigenvalues have positive real parts. This means that there exists a two-dimensional manifold in the four-dimensional phase-space which contains the steady state, such that if the solution to the modified Hamiltonian dynamical system starts on this manifold, it will asymptotically approach the steady state. Given (K_o, P_o) sufficiently close to the steady state, the initial values of λ_1 and λ_2 can be chosen so that the solution to the modified Hamiltonian dynamical system lies on this two-dimensional manifold and, thus, reaches the steady state asymptotically.

If the eigenvalues of J are real, then the solution to the modified Hamiltonian dynamical system that lies on the stable two-dimensional manifold converges asymptotically to the steady state as if it were a stable node. On the other hand, if the eigenvalues of J are complex, then the solution to the modified Hamiltonian dynamical system that lies on the stable two-dimensional manifold still converges asymptotically to the steady state, but now the optimal paths converge in a spiral fashion toward the steady state as if it were a stable focus. Note, in passing, that the sufficient condition we use to guarantee that the steady state is a local saddlepoint is not strong enough to determine if the eigenvalues of J are real or complex.

Our use of the stability condition follows the Revised Correspondence Principle of Brock, Malliaris (1989: Chapter 7), which states that the hypothesis of stability of the steady-state solution together with economically meaningful *a priori* structural assumptions on the integrand and state equations, lead to qualitatively useful steady-state comparative statics. In other words, Brock, Malliaris (1989) advocate, and then demonstrate, the importance of using sufficient conditions for stability of the steady-state solution in order to derive refutable steady-state comparative statics. We follow this methodological principle in that the sufficient condition $\Psi \geq 0$ for local saddlepoint stability of the steady-state solution is used to derive refutable steady-state comparative statics. Thus, the use of the stability hypothesis plays the same fundamental role as the maximization hypothesis, in that neither are directly observable, but both lead to useful qualitative restrictions on the observable variables and parameters of the underlying model.

Comparative statics of the steady state

Table 1 displays the steady-state comparative statics of the model. As the steady-state comparative statics for the parameter α_4 are ambiguous, they are omitted from the ensuing discussion and the table. To read Table 1, simply note that the symbol that occurs at the intersection of a given row and column (i.e. +, −, 0, ±) indicates the effect of an increase in the parameter in that column on the steady-state value of the variable in that row. Note that since all of the comparative statics pertain to the steady-state value of the variables, there is no need to continually repeat 'steady state' throughout this section.

First we examine the effects of a technology shock to the production function on the steady state of the economy. When the production technology parameter α_1 increases, the marginal and total product of capital increase by (A.3). The positive productivity shock leads the economy to accumulate more capital and, consequently, place a lower current value shadow price on the larger capital stock, just as predicted by the neo-classical optimal growth model. Such an inverse relationship between the capital stock and its current value shadow price is akin to the law of demand. Society, however, will *not* devote all additional income to consumption and capital investment, as is predicted by the neo-classical optimal growth model. It will, instead, choose to also increase abatement expenditures in order to reach a steady state that is deemed more acceptable in the presence of pollution. The effect of the productivity shock on the pollution stock and its current value shadow cost, however, is indeterminate, since consumption, the capital stock, and abatement are all higher in the new steady state. An application of the Dynamic Envelope Theorem of Caputo (1990) to (P) shows that the effect of an increase in the productivity of the capital stock on society's welfare is given by

$$\frac{\partial V(\theta)}{\partial \alpha_1} = \int_0^{+\infty} e^{-rt} \lambda_1^{\circ}(t;\theta) f_{\alpha_1}\left[K^{\circ}(t;\theta);\alpha_1\right] dt > 0$$

since $f_{\alpha_1} > 0$ by (A.3) and $\lambda_1^{\circ}(t;\theta) > 0$. Thus, the maximized present discounted value of social utility increases with a positive technology shock to the production function, and, hence, society is better off even if the steady-state stock of pollution is higher.

Table 1. Comparative statics of the steady state

Variables	Parameters					
	α_1	α_2	α_3	α_5	δ_2	r
K^*	+	0	0	0	0	−
P^*	±	+	+/0	−	±	±
λ_1^*	−	+	±	±	±	±
λ_2^*	±	−	+	+	+	±
A^*	+	−	−	±	−	−
C^*	+	±	+	±	+	±

An increase in the parameter α_2 represents a shift in preferences that causes consumption to be valued more highly at the margin by (A.2), i.e., the marginal utility of consumption is higher due to the increase in α_2. The increased marginal utility of consumption, surprisingly, has no effect on the capital stock and, therefore, on income, exactly as predicted by the neo-classical optimal growth model. The reason for this can be seen in the steady-state version of Equation (8), $f_K(K;\alpha_1) = r + \delta_1$, which shows that the capital stock is unaffected by any change in preferences, since neither the marginal cost nor the marginal product of capital are affected by a change in any parameter that enters the instantaneous utility function. Since capital is an input to production and thus consumption, the current value shadow price of capital increases due to the increase in the marginal utility of consumption, exactly as predicted by the neo-classical optimal growth model too. The pollution stock also increases since people now care relatively less about pollution, and, as a result, the shadow cost of pollution increases, that is, becomes more negative. An analogous calculation in Forster's (1973a) model (he did not actually perform such a calculation) similarly predicts the rise in pollution, but comes to the opposite conclusion with respect to its shadow cost. The increase in the marginal utility of consumption also results in a decrease in abatement, with an ambiguous change in consumption, however – the latter result contrasting sharply with the prediction of no change in consumption from the neo-classical optimal growth model and an increase in consumption from Forster's (1973a) model. Thus, the higher stock of pollution is due in part to the reduction in abatement, implying that even if consumption is lower, this does not offset the fall in abatement. By the Dynamic Envelope Theorem

$$\frac{\partial V(\theta)}{\partial \alpha_2} = \int_0^{+\infty} e^{-rt} U_{\alpha_2}\left[C^{\circ}(t;\theta);\alpha_2\right] > 0$$

since $U_{\alpha_2} > 0$ by (A.2). Thus, society's welfare increases when the marginal utility of consumption rises even though the steady-state stock of pollution is higher.

An increase in the parameter α_3 represents a preference shift in the marginal disutility of pollution by (A.2), whereby society now experiences less disutility from pollution present in the environment. When people derive less disutility from pollution, the capital stock and output are unaffected, since, as discussed above, the capital stock is unaffected by any change in preferences. The stock of pollution increases or remains unchanged as the same level of income is redistributed from abatement to consumption. A similar calculation in Forster's (1973a) model (again, he did not perform such a calculation) yields the same conclusion. Since the marginal disutility of pollution is now lower, so too is the shadow cost of pollution, which is also a conclusion that can be reached in Forster's (1973a) model. The effect of the shift in preference towards pollution on the shadow price of capital is, however, indeterminate. Society is better off when pollution is considered less harmful even though the steady-state stock of pollution is higher, for by the Dynamic Envelope Theorem

$$\frac{\partial V(\theta)}{\partial \alpha_3} = \int_0^{+\infty} e^{-rt} U_{\alpha_3}\left[P^{\circ}(t;\theta);\alpha_3\right] dt > 0,$$

since $U_{\alpha_3} > 0$ by (A.2).

An improvement in abatement technology is represented by a shift in the parameter α_5 which has no counterpart in Forster's (1973a) model nor in the neo-classical optimal growth model. By (A.5), an increase in α_5 signifies that more pollution can be abated in total and at the margin for a given amount of income devoted to abatement. When this technology shock occurs, the stock of pollution falls, and, consequently, so does the shadow cost of pollution. As in the previous discussions, the capital stock and income do not change. The current value shadow price of capital, however, could increase or decrease. This is due to the fact that abatement and consumption can increase or decrease as well. Unfortunately, in our four-dimensional dynamical system, we are unable to draw a phase diagram to examine the approach paths to the steady state. It is possible that, in the short run, abatement expenditures increase due to the increased efficiency of the abatement technology, and consumption expenditures decrease. This may cause less than optimal levels of pollution initially, so that in the long run, abatement expenditures will decrease and consumption will increase until a new steady state is reached. We can unambiguously conclude, however, that the maximized present discounted value of social utility increases with a positive shock to abatement technology, for by the Dynamic Envelope Theorem.

$$\frac{\partial V(\theta)}{\partial \alpha_5} \equiv -\int_0^{+\infty} e^{-rt}\lambda_2^o(t;\theta)h_{\alpha_5}\left[A^o(t;\theta);\alpha_5\right]dt > 0$$

since $h_{\alpha_5} > 0$ by (A.5) and $\lambda^o_2\ (t;\theta) < 0$.

Suppose that a country shifts production to industries with more biodegradable pollutant by-products, or that it uses biodegradable packaging for its consumable goods. Such a shift in production can be represented by an increase in the natural decay rate of pollution δ_2. When this parameter increases, the unchanged level of income is reallocated towards consumption and away from abatement as society takes advantage of the increase in nature's abatement productivity. Even so, the stock of pollution may be higher or lower in the new steady state. Forster (1973a) reached the same conclusion with respect to consumption and pollution. If the *ex ante* stock of pollution is 'small', then an increase in nature's ability to clean itself up will result in a larger *ex post* stock of pollution, perfectly consistent with lower abatement expenditures and higher consumption expenditures. On the other hand, if the *ex ante* stock of pollution is 'large', then the *ex post* stock of pollution will be smaller as a result of nature's increased ability to clean itself up. Such a result is seemingly inconsistent with the reallocation of income from abatement to consumption, but can be explained by a transitory phase in adjusting to the new steady state whereby abatement is higher and/or consumption is lower. Thus, in this instance, a policy reversal takes place in that the transitory effects on abatement and consumption due to the increase in δ_2 are the opposite of the steady-state effects. The marginal distillate of pollution in Equation (5) is now effectively discounted at a higher rate, causing the shadow cost of pollution to be lower, a result also predicted by Forster's (1973a) model. The effect on the shadow price of capital is, however, ambiguous. Society is better off due to the increase in nature's ability to reduce the stock of pollution even if the steady-state stock of pollution is larger, because by the Dynamic Envelope Theorem.

$$\frac{\partial V(\theta)}{\partial_{\delta_2}} \equiv -\int_0^{+\infty} e^{-rt}\lambda_2^o(t;\theta)P^o(t;\theta)dt > 0$$

since $\lambda_2^o\ (t;\theta) < 0$ and $P^o\ (t;\theta) > 0$.

Due to the introduction of pollution abatement as a control variable in our model, the steady-state comparative statics of an increase in the social discount rate differ considerably from those in a model without the abatement control variable. As expected, the capital stock and income decrease with an increase in the social discount rate, exactly as predicted by the neo-classical optimal growth model. Forster (1973a) found that consumption and pollution both increase when the social rate of discount increases, while, in the neo-classical optimal growth model, consumption falls. In contrast, our model shows that the effect of an increase in the social rate of discount on consumption and pollution is indeterminate once society is allowed to spend part of its income on pollution abatement. The level of abatement, however, decreases with the rise in the social discount rate.

Conclusion

The consideration to abate (and not just create) pollution is a natural and real choice. Recognizing this empirical fact, we have developed an optimal control model of a two-state economy to provide a better understanding of the trade-offs facing decision makers concerning their choices of consumption, capital accumulation, pollution accumulation, and abatement. While our model does not capture other important features that societies often possess, we have, at least, added one dimension to the prototype optimal growth and pollution control literature in an attempt to add a degree of realism.

Invoking a sufficient condition for local saddlepoint stability of the steady state, a comparative statics analysis of the steady state revealed the effects of preference and technology shocks on the economy's choice of consumption, abatement, the stocks of capital and pollution, and their current value shadow prices. Prototype steady-state comparative statics did not always hold because of the introduction of abatement as a decision variable in our model. Moreover, the welfare effects of these parameter shocks showed that, in some instances, society was better off even though the steady-state stock of pollution was larger.

Appendix I

The comparative statics of the optimal controls given in (9) are found by substituting (9a) into (2a) and (9b) into (2b) creating identities, and then differentiating the identities with respect to the parameter or variable of interest using the chain rule. Such a recipe yields:

$$\frac{\partial \hat{C}}{\partial \lambda_1} \equiv \frac{1}{H_{CC}} < 0,\ \frac{\partial \hat{C}}{\partial \lambda_2} \equiv -\frac{g_C}{H_{CC}} > 0,\ \frac{\partial \hat{C}}{\partial \alpha_2} \equiv -\frac{U_{C\alpha_2}}{H_{CC}} > 0,\ \frac{\partial \hat{C}}{\partial \alpha_4} \equiv -\frac{\lambda_2 g_{C\alpha_4}}{H_{CC}} > 0,$$

$$\frac{\partial \hat{A}}{\partial \lambda_1} \equiv \frac{1}{H_{AA}} < 0,\ \frac{\partial \hat{A}}{\partial \lambda_2} \equiv \frac{h_A}{H_{AA}} < 0,\ \frac{\partial \hat{A}}{\partial \alpha_5} \equiv \frac{\lambda_2 h_{A\alpha_5}}{H_{AA}} > 0,$$

where

$$H_{CC} \equiv U_{CC} + \lambda_2 g_{CC} < 0,\ H_{AA} \equiv -\lambda_2 h_{AA} < 0,$$

all terms are evaluated at

$$(C,A = [\hat{C}(\lambda_1,\lambda_2;\alpha_2,\alpha_4),\ \hat{A}(\lambda_1,\lambda_2;\alpha_5)],$$

and assumptions (A.2)–(A.5) along with the signs of the costate variables were used to sign the above comparative statics. The remaining comparative statics of

$$(C,A) = [\hat{C}(\lambda_1,\lambda_2;\alpha_2,\alpha_4),\ \hat{A}(\lambda_1,\lambda_2;\alpha_5)]$$

for the variables and parameters which they are not a function of are identically zero.

Appendix II

The elements of the Jacobian matrix (11) of the Modified Hamiltonian Dynamical System (10) are given by:

$$\frac{\partial \dot{K}}{\partial K} = f_K - \delta_1 \equiv r > 0,\ \frac{\partial \dot{K}}{\partial P} = 0,\ \frac{\partial \dot{K}}{\partial \lambda_1} = -\frac{\partial \hat{C}}{\partial \lambda_1} - \frac{\partial \hat{A}}{\partial \lambda_1} > 0,\ \frac{\partial \dot{K}}{\partial \lambda_2} = -\frac{\partial \hat{C}}{\partial \lambda_2} - \frac{\partial \hat{A}}{\partial \lambda_2} >< 0,$$

$$\frac{\partial \dot{P}}{\partial K} = 0,\ \frac{\partial \dot{P}}{\partial P} = -\delta_2 < 0,\ \frac{\partial \dot{P}}{\partial \lambda_1} = g_C \frac{\partial \hat{C}}{\partial \lambda_1} - h_A \frac{\partial \hat{A}}{\partial \lambda_1} = \frac{\partial \dot{K}}{\partial \lambda_2} >< 0,\ \frac{\partial \dot{P}}{\partial \lambda_2} = g_C \frac{\partial \hat{C}}{\partial \lambda_2} - h_A \frac{\partial \hat{A}}{\partial \lambda_2} > 0,$$

$$\frac{\partial \dot{\lambda}_1}{\partial K} = -\lambda_1 f_{KK} > 0,\ \frac{\partial \dot{\lambda}_1}{\partial P} = 0,\ \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} = 0,\ \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} = 0,$$

$$\frac{\partial \dot{\lambda}_2}{\partial K} = 0,\ \frac{\partial \dot{\lambda}_2}{\partial P} = -U_{PP} > 0,\ \frac{\partial \dot{\lambda}_2}{\partial \lambda_1} = 0,\ \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} = r + \delta_2 > 0,$$

where all the terms are evaluated at the steady state given by (12), and the results of Appendix I and assumptions (A.2)–(A.5) were used to sign the derivatives and to prove that $\partial \dot{P}/\partial \lambda_1 = \partial \dot{K}/\partial \lambda_2$, a result used many times to help sign the steady-state comparative statics in Appendix III.

Appendix III

The comparative statics of the steady state are found by substituting the steady-state solution (12) into the steady-state version of (10) to create identities in β, differentiating the identities with respect to the parameter of interest using the chain rule, evaluating the result at the steady state, and solving the resulting linear system with Cramer's rule. Following that recipe for α_1 yields:

$$\frac{\partial K^*}{\partial \alpha_1} \equiv \frac{f_{K\alpha_1}}{f_{KK}} > 0, \quad \frac{\partial P^*}{\partial \alpha_1} \equiv \frac{-\frac{\partial \dot{P}}{\partial \lambda_1}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}\left(\lambda_1 f_{K\alpha_1}\frac{\partial \dot{K}}{\partial K} + f_{\alpha_1}\frac{\partial \dot{\lambda}_1}{\partial K}\right)}{|J|} \gtrless 0,$$

$$\frac{\partial \lambda_1^*}{\partial \alpha_1} \equiv \frac{\left(\lambda_1 f_{K\alpha_1}\frac{\partial \dot{K}}{\partial K} + f_{\alpha_1}\frac{\partial \dot{\lambda}_1}{\partial K}\right)\left(\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} - \frac{\partial \dot{P}}{\partial \lambda_2}\frac{\partial \dot{\lambda}_2}{\partial P}\right)}{|J|} < 0,$$

$$\frac{\partial \lambda_2^*}{\partial \alpha_1} \equiv \frac{\frac{\partial \dot{P}}{\partial \lambda_1}\frac{\partial \dot{\lambda}_2}{\partial P}\left(\lambda_1 f_{K\alpha_1}\frac{\partial \dot{K}}{\partial K} + f_{\alpha_1}\frac{\partial \dot{\lambda}_1}{\partial K}\right)}{|J|} \gtrless 0,$$

The steady-state comparative statics for α_2 are given by:

$$\frac{\partial K^*}{\partial \alpha_2} \equiv 0, \quad \frac{\partial P^*}{\partial \alpha_2} \equiv \frac{\frac{\partial \hat{C}}{\partial \alpha_2}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}\left(\frac{\partial \dot{P}}{\partial \lambda_1} + g_C\frac{\partial \dot{K}}{\partial \lambda_1}\right)}{|J|} > 0,$$

$$\frac{\partial \lambda_1^*}{\partial \alpha_2} \equiv \frac{-\frac{\partial \hat{C}}{\partial \alpha_2}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} + \frac{\partial \hat{C}}{\partial \alpha_2}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial P}\left(\frac{\partial \dot{P}}{\partial \lambda_2} + g_C\frac{\partial \dot{K}}{\partial \lambda_2}\right)}{|J|} > 0$$

$$\frac{\partial \lambda_2^*}{\partial \alpha_2} \equiv \frac{-\frac{\partial \hat{C}}{\partial \alpha_2}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial P}\left(\frac{\partial \dot{P}}{\partial \lambda_1} + g_C\frac{\partial \dot{K}}{\partial \lambda_1}\right)}{|J|} > 0,$$

where

$$\left(\frac{\partial \dot{P}}{\partial \lambda_1} + g_C\frac{\partial \dot{K}}{\partial \lambda_1}\right) = -(h_A + g_C)\frac{\partial \hat{A}}{\partial \lambda_1} > 0 \text{ and } \left(\frac{\partial \dot{P}}{\partial \lambda_2} + g_C\frac{\partial \dot{K}}{\partial \lambda_2}\right) = -(h_A + g_C)\frac{\partial \hat{A}}{\partial \lambda_2} > 0$$

follow from Appendices I and II.

The steady-state comparative statics for α_3 are given by:

$$\frac{\partial K^*}{\partial \alpha_3} \equiv 0, \quad \frac{\partial P^*}{\partial \alpha_3} \equiv \frac{U_{P\alpha_3}\frac{\partial \dot{\lambda}_1}{\partial K}\Psi}{|J|} \geq 0, \quad \frac{\partial \lambda_1^*}{\partial \alpha_3} \equiv \frac{U_{P\alpha_3}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_2}\frac{\partial \dot{P}}{\partial P}}{|J|} \gtrless 0, \quad \frac{\partial \lambda_2^*}{\partial \alpha_3} \equiv \frac{-U_{P\alpha_3}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_1}\frac{\partial \dot{P}}{\partial P}}{|J|} > 0.$$

The steady-state comparative statics for α_5 are given by:

$$\frac{\partial K^*}{\partial \alpha_5} \equiv 0, \quad \frac{\partial P^*}{\partial \alpha_5} \equiv \frac{\frac{\partial \hat{A}}{\partial \alpha_5}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}\left(\frac{\partial \dot{P}}{\partial \lambda_1} - h_A\frac{\partial \dot{K}}{\partial \lambda_1}\right) - h_{\alpha_5}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_1}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}}{|J|} < 0,$$

$$\frac{\partial \lambda_1^*}{\partial \alpha_5} \equiv \frac{\frac{\partial \hat{A}}{\partial \alpha_5}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial P}\left(\frac{\partial \dot{P}}{\partial \lambda_2} - h_A\frac{\partial \dot{K}}{\partial \lambda_2}\right) - \frac{\partial \dot{\lambda}_1}{\partial K}\left(h_{\alpha_5}\frac{\partial \dot{K}}{\partial \lambda_2}\frac{\partial \dot{\lambda}_2}{\partial P} + \frac{\partial \hat{A}}{\partial \alpha_5}\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}\right)}{|J|} \gtrless 0,$$

$$\frac{\partial \lambda_2^*}{\partial \alpha_5} \equiv \frac{-\frac{\partial \hat{A}}{\partial \alpha_5}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial P}\left(\frac{\partial \dot{P}}{\partial \lambda_1} - h_A\frac{\partial \dot{K}}{\partial \lambda_1}\right) + h_{\alpha_5}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_1}\frac{\partial \dot{\lambda}_2}{\partial P}}{|J|} > 0,$$

where

$$\left(\frac{\partial \dot{P}}{\partial \lambda_1} - h_A\frac{\partial \dot{K}}{\partial \lambda_1}\right) = (h_A + g_C)\frac{\partial \hat{C}}{\partial \lambda_1} < 0 \text{ and } \left(\frac{\partial \dot{P}}{\partial \lambda_2} - h_A\frac{\partial \dot{K}}{\partial \lambda_2}\right) = (h_A + g_C)\frac{\partial \hat{C}}{\partial \lambda_2} > 0$$

follow from Appendices I and II.

The steady-state comparative statics for δ_2 are given by:

$$\frac{\partial K^*}{\partial \delta_2} \equiv 0, \quad \frac{\partial P^*}{\partial \delta_2} \equiv \frac{-\frac{\partial \dot{\lambda}_1}{\partial K}\left(\lambda_2\Psi + P\frac{\partial \dot{K}}{\partial \lambda_1}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}\right)}{|J|} \gtrless 0,$$

$$\frac{\partial \lambda_1^*}{\partial \delta_2} \equiv \frac{-\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_2}\left(\lambda_2\frac{\partial \dot{P}}{\partial P} + P\frac{\partial \dot{\lambda}_2}{\partial P}\right)}{|J|} \gtrless 0, \quad \frac{\partial \lambda_2^*}{\partial \delta_2} \equiv \frac{\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_1}\left(\lambda_2\frac{\partial \dot{P}}{\partial P} + P\frac{\partial \dot{\lambda}_2}{\partial P}\right)}{|J|} > 0.$$

The steady-state comparative statics for r are given by:

$$\frac{\partial K^*}{\partial r} = \frac{\lambda_1 \frac{\partial \dot{K}}{\partial \lambda_1}\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} - \lambda_1 \frac{\partial \dot{\lambda}_2}{\partial P}\Psi}{|J|} < 0, \quad \frac{\partial P^*}{\partial r} = \frac{\lambda_1 \frac{\partial \dot{K}}{\partial K}\frac{\partial \dot{P}}{\partial \lambda_1}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} - \lambda_2 \frac{\partial \dot{\lambda}_1}{\partial K}\Psi}{|J|} >< 0,$$

$$\frac{\partial \lambda_1^*}{\partial r} = \frac{\lambda_1 \frac{\partial \dot{K}}{\partial K}\left(\frac{\partial \dot{P}}{\partial \lambda_2}\frac{\partial \dot{\lambda}_2}{\partial P} - \frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}\right) - \lambda_2 \frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{K}}{\partial \lambda_2}\frac{\partial \dot{P}}{\partial P}}{|J|} >< 0,$$

$$\frac{\partial \lambda_2^*}{\partial r} = \frac{-\lambda_1 \frac{\partial \dot{K}}{\partial K}\frac{\partial \dot{P}}{\partial \lambda_1}\frac{\partial \dot{\lambda}_2}{\partial P} + \lambda_2 \frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{K}}{\partial \lambda_1}}{|J|} >< 0.$$

The steady-state comparative statics of the optimal controls are found by differentiating (13) using the chain rule, evaluating the result at the steady state, and using the results of Appendices I, II, and III. Following this recipe for steady-state consumption gives:

$$\frac{\partial C^*}{\partial \alpha_1} = \frac{\left(\lambda_1 f_{K\alpha_1}\frac{\partial \dot{K}}{\partial K} + f_{\alpha_1}\frac{\partial \dot{\lambda}_1}{\partial K}\right)\left(\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} - \frac{\partial \dot{\lambda}_2}{\partial P}\left[\frac{\partial \dot{P}}{\partial \lambda_2} + g_C \frac{\partial \dot{K}}{\partial \lambda_2}\right]\right)}{H_{CC}|J|} > 0,$$

$$\frac{\partial C^*}{\partial \alpha_2} = \frac{\partial \hat{C}}{\partial \lambda_1}\frac{\partial \lambda_1^*}{\partial \alpha_2} + \frac{\partial \hat{C}}{\partial \lambda_2}\frac{\partial \lambda_2^*}{\partial \alpha_2} + \frac{\partial \hat{C}}{\partial \alpha_2} >< 0,$$

$$\frac{\partial C^*}{\partial \alpha_3} = \frac{U_{P\alpha_3}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{P}}{\partial P}\left(\frac{\partial \dot{P}}{\partial \lambda_1} + g_C \frac{\partial \dot{K}}{\partial \lambda_1}\right)}{H_{CC}|J|} > 0,$$

$$\frac{\partial C^*}{\partial \alpha_5} = \frac{\partial \hat{C}}{\partial \lambda_1}\frac{\partial \lambda_1^*}{\partial \alpha_5} + \frac{\partial \hat{C}}{\partial \lambda_2}\frac{\partial \lambda_2^*}{\partial \alpha_5} >< 0,$$

$$\frac{\partial C^*}{\partial \delta_2} = \frac{-\frac{\partial \dot{\lambda}_1}{\partial K}\left(\lambda_2 \frac{\partial \dot{P}}{\partial P} + P\frac{\partial \dot{\lambda}_2}{\partial P}\right)\left(\frac{\partial \dot{P}}{\partial \lambda_1} + g_C \frac{\partial \dot{K}}{\partial \lambda_1}\right)}{H_{CC}|J|} > 0,$$

$$\frac{\partial C^*}{\partial r} = \frac{\partial \hat{C}}{\partial \lambda_1}\frac{\partial \lambda_1^*}{\partial r} + \frac{\partial \hat{C}}{\partial \lambda_2}\frac{\partial \lambda_2^*}{\partial r} >< 0,$$

where the results of Appendices II and III were used to simplify and sign the expressions. Finally, the steady-state comparative statics of abatement are given by:

$$\frac{\partial A^*}{\partial \alpha_1} = \frac{\left(\lambda_1 f_{K\alpha_1}\frac{\partial \dot{K}}{\partial K} + f_{\alpha_1}\frac{\partial \dot{\lambda}_1}{\partial K}\right)\left(\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2} - \frac{\partial \dot{\lambda}_2}{\partial P}\left[\frac{\partial \dot{P}}{\partial \lambda_2} - h_A \frac{\partial \dot{K}}{\partial \lambda_1}\right]\right)}{H_{AA}|J|} > 0,$$

$$\frac{\partial A^*}{\partial \alpha_2} = \frac{-\frac{\partial \hat{C}}{\partial \alpha_2}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}}{H_{AA}|J|} < 0,$$

$$\frac{\partial A^*}{\partial \alpha_3} = \frac{U_{P\alpha_3}\frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{P}}{\partial P}\left(\frac{\partial \dot{P}}{\partial \lambda_1} - h_A \frac{\partial \dot{K}}{\partial \lambda_1}\right)}{H_{AA}|J|} < 0,$$

$$\frac{\partial A^*}{\partial \alpha_5} = \frac{\partial \hat{A}}{\partial \lambda_1}\frac{\partial \lambda_1^*}{\partial \alpha_5} + \frac{\partial \hat{A}}{\partial \lambda_2}\frac{\partial \lambda_2^*}{\partial \alpha_5} + \frac{\partial \hat{A}}{\partial \alpha_5} >< 0,$$

$$\frac{\partial A^*}{\partial \delta_2} = \frac{-\frac{\partial \dot{\lambda}_1}{\partial K}\left(\lambda_2 \frac{\partial \dot{P}}{\partial P} + P\frac{\partial \dot{\lambda}_2}{\partial P}\right)\left(\frac{\partial \dot{P}}{\partial \lambda_1} - h_A \frac{\partial \dot{K}}{\partial \lambda_1}\right)}{H_{AA}|J|} < 0,$$

$$\frac{\partial A^*}{\partial r} = \frac{\lambda_1 \frac{\partial \dot{K}}{\partial K}\frac{\partial \dot{\lambda}_2}{\partial P}\left(\frac{\partial \dot{P}}{\partial P} - h_A \frac{\partial \dot{K}}{\partial \lambda_2}\right) - \lambda_2 \frac{\partial \dot{\lambda}_1}{\partial K}\frac{\partial \dot{P}}{\partial P}\left(\frac{\partial \dot{P}}{\partial \lambda_1} - h_A \frac{\partial \dot{K}}{\partial \lambda_1}\right) - \lambda_1 \frac{\partial \dot{K}}{\partial K}\frac{\partial \dot{P}}{\partial P}\frac{\partial \dot{\lambda}_2}{\partial \lambda_2}}{H_{AA}|J|} < 0,$$

where the results of Appendices II and III were used to simplify and sign the expressions.

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The modelling of the petroleum exploration and extraction process for policy analysis: a case study of the Louisiana onshore region

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This paper presents a region-specific analytical model of petroleum exploration, development, and extraction. The model is estimated and simulated to investigate the effects of natural gas and crude oil prices, severance and royalty taxes, and the corporate tax rate on petroleum drilling, new reserve additions, and production in onshore Louisiana. Our simulation results suggest that drilling, reserve additions, and the production of oil and gas have relatively low sensitivity to changes in severance oil and gas tax rates. In addition, we found that drilling, reserve additions, and petroleum production in onshore Louisiana are relatively price- and tax-inelastic. We also found that the response of drilling, reserve additions, and production in South Louisiana, though equally inelastic with respect to prices, corporate taxes, and royalty, is generally more sensitive to changes in petroleum drilling determinants than they are in North Louisiana.

Introduction

The majority of models developed to analyse petroleum supply in the US following the 1973 oil embargo suffer from a number of shortcomings. Specifically, these models are either pure geological and engineering models which exclude the effects of economic factors on oil and gas supply, or econometric models which do not account for engineering, geophysical, and geological factors affecting petroleum resource development. In addition, these modelling frameworks are national in scope, and tend to obscure the regional characteristics that are unique to individual petroleum basins. Thus, neither of these modelling approaches have performed satisfactorily in explaining or predicting how the firms in the US oil and gas industry explore for new reserves, develop reserve additions, and produce oil and gas reserves.

More recently, however, region-specific models of petroleum exploration and extraction have been developed for the states of California (Deacon *et al.* 1990), and West Virginia (Iledare 1990). Furthermore, a hybrid modelling framework has been espoused for analysing petroleum exploration and extraction behaviour in direct response to the shortcomings of a pure geological and engineering model or econometric models (Walls 1992). Such a model framework was used to describe the oil and gas supply process in coastal Louisiana (Dupont 1993), and the Gulf of Mexico Outer Continental Shelf (OCS) (Walls 1994).

The analytical model presented in this paper is a region-specific hybrid model. The model framework assumes that the fundamental driving force underlying exploration and reserve development effort is profit maximization, subject to a diminishing rate of reserve